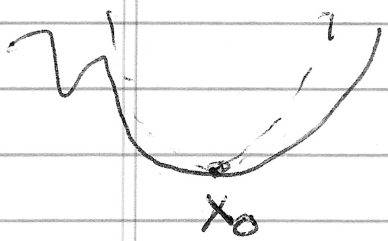
Ponderomotive Force

Ponderomotive Force

→ Motivation

Recall inverted pendulum movie.

- consider motion in potential $U(x, \lambda)$.



Then, particle will sit near minimum, so

$$U = U_0 + \frac{1}{2} M \Omega_0^2 (x - x_0)^2$$

⇒ motion has natural frequency Ω_0

$$U'' = \frac{1}{2} M \Omega_0^2 \Big|_{x_0}$$

- now, $\lambda = \lambda(t)$

$\lambda(t)$:

- $\dot{\lambda}/\lambda \ll \Omega_0$ ⇒ adiabatic limit leading to adiabatic invariants
- $\dot{\lambda}/\lambda \gg \Omega_0$:
  - ponderomotive force
  - parametric instability $\dot{\lambda}/\lambda \sim 2\Omega_0$

- consider 1D problem with high frequency driver f

i.e.

$$m\ddot{x} = -\frac{dU}{dx} + \underbrace{f}_{\text{"small"}}$$

→ high frequency drive.

→ Recall movie of inverted pendulum:

- what happened?
- why is it stable while inverted?
- is there a criticality condition? what sets it?
- How to approach theoretically?
- what leverage (small parameter) do we have on the problem?

$\boxed{\text{idea: driver on the pendulum}}$

now $F = f_1 \cos \omega t + f_2 \sin \omega t$

$$= F_1(x) \cos \omega t + F_2(x) \sin \omega t$$

$\omega \gg \Omega_0$

$\cos \omega t$, $\sin \omega t$ → fast variation on time

$F_1(x)$, $F_2(x)$ → spatial envelope → position dependent

→ idea is that f_1, f_2 induce oscillation on motion in U.

Can we describe "mean motion"?

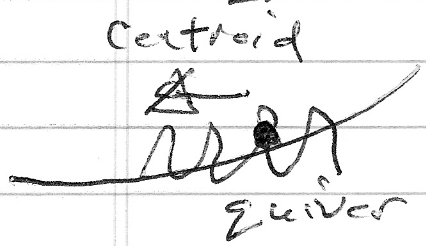
⇒ Primary interest in centroid motion...

so

$$x(t) = y(t) + \xi(t)$$

$y(t)$ → slow centroid motion [quiver center]

$\xi(t)$ ↳ fast quivering motion on frequency ω [quiver]



Now, $x = y + \xi$ separates motion into slow, fast components
into

$y \to$ slow
$\xi \to$ fast

$\boxed{\text{Time/Frequency scale separation is key.}}$

Now, since ξ tracks f:

$$\tau = 2\pi/\omega \; ; \qquad \underbrace{\frac{1}{\tau}\int_0^{2\pi/\omega} dt\, \xi(t) = 0}.$$

period average of high frequency oscillation vanishes.

So, EOM:

$$m\ddot{x} = -\frac{dU}{dx} + f$$

$\Rightarrow$ and $\quad x = y + \xi$

$$m(\ddot{y} + \ddot{\xi}) = -\frac{d}{dx}(U(y+\xi)) + f(y+\xi)$$

now expand:

(assumes ξ small, no resonances)

$$m(\ddot{y} + \ddot{\varepsilon}) = -\frac{d}{dx}U(x)\Big|_y - \varepsilon \frac{d^2U}{dx^2}\Big|_y + f(y) + \varepsilon \frac{df}{dx}\Big|_y$$

note this is a <u>beat</u>

$\varepsilon \sim \cos\omega t$

$df/dx \sim \cos\omega t$

$\Rightarrow$ generates <u>low</u> frequency!

$\Rightarrow$

$$m(\ddot{y} + \ddot{\varepsilon}) = \overset{①}{-\frac{dU(y)}{dy}} \; \overset{②}{- \varepsilon \frac{d^2U(y)}{dy^2}} \; \overset{③}{+ f(y)} \; \overset{④}{+ \varepsilon \frac{df}{dy}}.$$

① $\rightarrow$ slow time scale

② $\rightarrow$ fast time scale

③ $\rightarrow$ fast time scale ($\sim \cos\omega t$)

④ $\rightarrow$ <u>slow</u> (via beat).

Now, to separate:

$$\langle \ \rangle = \frac{1}{\tau}\int_0^\tau dt$$

so

$$\overset{S}{m\langle \ddot{y} \rangle} + \overset{F}{m\langle \ddot{\tilde{\varepsilon}} \rangle} = -\overset{S}{\left\langle \frac{dU}{dy} \right\rangle}$$

$$- \left\langle \overset{F}{\tilde{\varepsilon}} \overset{S}{\frac{d^2U}{dy^2}} \right\rangle + \overset{F}{\langle f \rangle} + \overset{S}{\left\langle \tilde{\varepsilon} \frac{\partial f}{\partial y} \right\rangle}$$

$$\langle \ddot{\tilde{\varepsilon}} \rangle = 0$$
$$\langle \tilde{\varepsilon} \rangle = 0$$

$$\boxed{m\ddot{y} = -\frac{dU}{dy} + \left\langle \tilde{\varepsilon} \frac{\partial f}{\partial y} \right\rangle}$$

non-trivial survivor

slow, or mean field, eqn.

For fast,

$$m\ddot{\tilde{\varepsilon}} = -\tilde{\varepsilon}\frac{d^2U}{dy^2} + f(y)$$

but: $\quad m\ddot{\tilde{\xi}} \sim m\omega^2\tilde{\xi}$

$$\tilde{\xi}\frac{d^2 U}{dy^2} \sim m\Omega_0^2\tilde{\xi}$$

and $\omega^2 \gg \Omega_0^2$

so

$$\boxed{m\ddot{\tilde{\xi}} = f(y)} \rightarrow \text{fast eqn.}$$

proceed:

ⓐ – solve fast eqn.
ⓑ – iterate into slow ⇒ calculate best
ⓒ – solve slow

Now;

ⓐ $m\ddot{\tilde{\xi}} = f_1(y)\cos\omega t + f_2(y)\sin\omega t$

$$\tilde{\xi} = \frac{-f(y)}{m\omega^2}$$ → fast quiver variation (f is space, time dependent; time dependence is slow)

then, for beat:

$$\text{beat} = (4) = \left\langle \tilde{\varepsilon} \frac{\partial f}{\partial y} \right\rangle$$

$$= \left\langle \frac{-f}{m\omega^2} \frac{\partial f}{\partial y} \right\rangle$$

$\Rightarrow$

$$m\ddot{y} = -\frac{dU}{dy}$$ [redacted] $$- \frac{1}{m\omega^2} \left\langle f \frac{\partial f}{\partial y} \right\rangle$$

here:

$$f = f_1 \cos\omega t + f_1 \sin\omega t$$

$$\langle\,\rangle = \frac{1}{\tau} \int_0^\tau dt$$

$\Rightarrow$

$$m\ddot{y} = -\frac{dU}{dy} - \frac{1}{2m\omega^2} \left\{ f_1 \frac{\partial f_1}{\partial y} + f_2 \frac{\partial f_2}{\partial y} \right\}$$

(cross terms gone)

$$m\ddot{y} = -\frac{dU}{dy} - \frac{1}{2m\omega^2} \frac{d}{dy} \left\{ f^2 \right\}$$

$$\{f^2\} = \frac{1}{2}(f_1^2 + f_2^2)$$

$\Rightarrow$ 6

$$\boxed{m\ddot{y} = -\frac{d}{dy}\left[u(y) + \frac{1}{4m\omega^2}(f_1^2(y) + f_2^2(y))\right]}$$

EOM for y $\Leftrightarrow$ mean field eqn.

Effective potential becomes:

$$\boxed{U_{eff} = u(y) + \frac{1}{4m\omega^2}(f_1^2(y) + f_2^2(y))}$$

where ponderomotive potential is:

$$\boxed{U_{pondero} = \frac{1}{4m\omega^2}\left(f_1^2 + f_2^2\right)}$$

is piece induced by high frequency quiver.

then,

$$\boxed{\begin{aligned} f_{ponderomotive} &= -\nabla U_{pondero} \\ &= \frac{-1}{4m\omega^2}\frac{d}{dy}\left(f_1^2+f_2^2\right) \end{aligned}}$$

- ponderomotive force
- ~ gradient of spatial envelope of high frequency energy field
- observe:

$$\frac{1}{4m\omega^2}\left(f_1^2+f_2^2\right) = \frac{1}{2}m\langle\dot{\tilde{\xi}}^2\rangle$$

quiver kinetic energy

alternatively,

$$\boxed{U_{ponderomotive} = \frac{1}{2}m\langle\dot{\tilde{\xi}}^2\rangle}$$

where:

$$\boxed{m\ddot{\tilde{\xi}} = f}$$

→ effective kinetic energy field of quiver kinetic energy

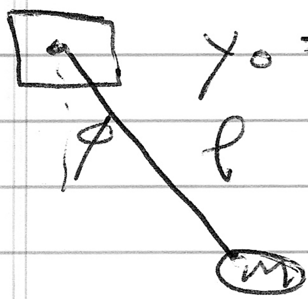
→ gradient in kinetic energy field induces mean force

Note: Wave momentum theorem

$$\frac{dP_w}{dt} = -\nabla \mathcal{H} = -\nabla \mathcal{E} \qquad (\partial_t \mathcal{L} = 0)$$

has character of ponderomotives. Gradient of energy density exerts force!.

→ (Classic) Example - Inverted Pendulum (recall movie)



$y_0 = y_0 \cos \omega t$ $\qquad \omega >> g/\ell$

$y = y_0 \cos \omega t + \ell \cos \phi$

$x = \ell \sin \phi$

Consider "the usual" pendulum with vertically oscillating point of support.

$$L = T - U$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgl(1 - \cos\phi)$$

$$L = \frac{1}{2} m \left(l^2 \cos^2\phi \, \dot{\phi}^2 + \left(\omega y_0 \sin\omega t + l \sin\phi \, \dot{\phi}\right)^2\right)$$

$$- mgl(1 - \cos\phi)$$

$$= \frac{1}{2} m \Big(\left(l^2 \cos^2\phi + l^2 \sin^2\phi\right)\dot{\phi}^2$$

$$+ 2\omega y_0 l \dot{\phi} \sin\omega t \sin\phi + \omega^2 y_0^2 \sin^2\omega t \Big)$$

h.o. on y_0
$\Rightarrow$ weak
(ϕ small)

$\therefore$

$$L = \frac{1}{2} m \left(l^2 \dot{\phi}^2 + 2\omega y_0 l \dot{\phi} \sin\phi \sin\omega t\right)$$

$$+ mgl \cos\phi$$

oscillating term

LEOM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$\Rightarrow$

$$\frac{d}{dt}\left(m\ell^2\dot{\phi} + m\omega y_0 \ell \sin\omega t \sin\phi\right)$$

$$= -mg\ell\sin\phi + m\omega y_0 \ell\dot{\phi}\sin\omega t\cos\phi$$

$\Rightarrow$

$$m\ell^2\ddot{\phi} + m\omega y_0 \ell\left(\omega\cos\omega t\sin\phi + \sin\omega t\cos\phi\dot{\phi}\right)$$

$$= -mg\ell\sin\phi + m\omega y_0\ell\dot{\phi}\sin\omega t\cos\phi$$

$$\boxed{m\ell^2\ddot{\phi} = -mg\ell\sin\phi - m\omega^2 y_0\ell\sin\phi\cos\omega t}$$

$$\boxed{\ddot{\phi} = -\frac{g}{\ell}\sin\phi - \omega^2\frac{y_0}{\ell}\sin\phi\cos\omega t}$$

in form:

$$\ddot{\phi} = -\frac{d\,U(\phi)}{d\phi} + f_1(\phi)\cos\omega t$$

$\Longleftrightarrow$

$$\ddot{\phi} = -\frac{g}{\ell}\sin\phi - \frac{\omega^2 y_0}{\ell}\sin\phi\cos\omega t$$

$\underline{\underline{so}}$

$$U(\phi) = \frac{g}{\ell}\cos\phi$$

$\omega \gg \sqrt{g/\ell}$

$$f_1(\phi) = -\frac{\omega^2 y_0}{\ell}\sin\phi$$

To solve:

$$\phi = \langle\phi\rangle + \tilde{\phi}$$

$$= \underbrace{\bar{\phi}}_{\text{notational brevity}} + \tilde{\phi}$$

$$\ddot{\bar{\phi}} + \ddot{\tilde{\phi}} = -\frac{g}{\ell}\left(\sin\left[\bar{\phi}+\tilde{\phi}\right]\right) - \frac{\omega^2 y_0}{\ell}\sin\left(\bar{\phi}+\tilde{\phi}\right)\cos\omega t$$

expanding $\Rightarrow$

$$\ddot{\bar{\phi}} + \ddot{\tilde{\phi}} = \overset{①}{-\frac{g}{\ell}\sin\bar{\phi}} \quad \overset{②}{-\tilde{\phi}\frac{g}{\ell}\cos\bar{\phi}}$$

$$\overset{③}{-\omega^2\frac{y_0}{\ell}\sin\bar{\phi}\cos\omega t} \quad \overset{④}{-\omega^2\frac{y_0}{\ell}\cos\bar{\phi}\,\tilde{\phi}\cos\omega t}$$

then average:

$$\langle ② \rangle = \langle ③ \rangle = 0$$

$$\ddot{\bar{\phi}} = -\frac{g}{\ell}\sin\bar{\phi} - \omega^2\frac{y_0}{\ell}\cos\bar{\phi}\langle\tilde{\phi}\cos\omega t\rangle$$

To compute $\langle\ \rangle$, need $\tilde{\phi}$

$$\ddot{\tilde{\phi}} = \cancel{-\tilde{\phi}\frac{g}{\ell}\cos\bar{\phi}} - \omega^2\frac{y_0}{\ell}\sin\bar{\phi}\cos\omega t$$

$g/\ell \ll \omega^2$

$$\tilde{\phi} = a(\bar{\phi})\cos\omega t$$

$$-\omega^2 a(\bar{\phi})\cos\omega t = -\omega^2 \frac{y_0}{\ell}\sin\bar{\phi}\,\cos\omega t$$

$$a(\bar{\phi}) = \frac{y_0}{\ell}\sin\bar{\phi}$$

$$\langle \phi\cos\omega t\rangle = \left\langle \frac{y_0}{\ell}\sin\bar{\phi}\,\cos^2\omega t\right\rangle$$

$$= \frac{y_0}{2\ell}\sin\bar{\phi}$$

so

$$\langle (4)\rangle = -\frac{\omega^2 y_0^2}{2\ell^2}\sin\bar{\phi}\cos\bar{\phi}$$

$$= -\frac{\omega^3 y_0^2}{4\ell^2}\sin 2\bar{\phi}$$

so $\Rightarrow$

$$\boxed{\ddot{\bar{\phi}} = -\frac{g}{\ell}\sin\bar{\phi} - \frac{\omega^2 y_0^2}{4\ell^2}\sin 2\bar{\phi}}$$

$$-\omega^2 a(\bar{\phi})\cos\omega t = -\omega^2 \frac{y_0}{\ell}\sin\bar{\phi}\cos\omega t$$

$$a(\bar{\phi}) = \frac{y_0}{\ell}\sin\bar{\phi}$$

$$\langle\tilde{\phi}\cos\omega t\rangle = \frac{y_0}{\ell}\langle\sin\bar{\phi}\cos^2\omega t\rangle$$

$$= \frac{y_0}{2\ell}\sin\bar{\phi}$$

so

$$\langle 4 \rangle = -\frac{\omega^2 y_0^2}{2\ell^2}\sin\bar{\phi}\cos\bar{\phi}$$

$$= -\frac{\omega^2 y_0^2}{4\ell^2}\frac{d}{d\phi}\sin^2\bar{\phi}$$

so

$$\ddot{\bar{\phi}} = -\frac{d}{d\phi}\left(-\frac{g}{\ell}\cos\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2}\sin^2\bar{\phi}\right)$$

$$\boxed{\begin{aligned} U_{eff} &= U_0 + U_{pond} \\ U_0 &= -\frac{g}{\ell} \cos\bar{\phi} \\ U_{pond} &= \frac{\omega^2 y_0^2}{4\ell^2} \sin^2\bar{\phi} \end{aligned}}$$

effective potential.

Now, with inverted pendulum in mind, look for minima (1D)!

$$U(\bar{\phi}) = -\frac{g}{\ell}\cos\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2}\sin^2\bar{\phi}$$

$$= -\frac{g}{\ell}\cos\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2}\left(\frac{1}{2} - \frac{1}{2}\cos 2\bar{\phi}\right)$$

extrema:

$$\frac{dU}{d\bar{\phi}} = \frac{g}{\ell}\sin\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2}\sin 2\bar{\phi} = 0$$

$\bar{\phi} = 0$ is minimum

$\bar{\phi} = \pi$ is extremum.

Now,

$$\frac{d^2 U}{d\phi^2} = \frac{g}{\ell}\cos\bar{\phi} + \frac{\omega^2 y_0^2}{2\ell^2}\cos 2\bar{\phi}$$

for $\bar{\phi} = \pi$

$$= -\frac{g}{\ell} + \frac{\omega^2 y_0^2}{2\ell^2}$$

$\Rightarrow$ $\left\{ \begin{array}{l} \text{extremum can be} \\ \text{minimum for} \\ y_0^2\omega^2/2\ell^2 > g/\ell. \end{array} \right.$

so for $\frac{\omega^2 y_0^2}{2\ell^2} > \frac{g}{\ell}$, $\bar{\phi} = \pi$ is a (stable) minimum.

$\Rightarrow$

- pendulum effective potential has second minimum at inversion ($\bar{\phi} = \pi$) for $\boxed{\frac{\omega^2 y_0^2}{2\ell^2} > \frac{g}{\ell}}$

- establishes critical ponderomotive potential strength.

Comments:

$\rightarrow$ have encountered ponderomotive force i.e. gradient on spatially varying envelope of high frequency energy field

→ can think of ponderomotive potential as effective radiation pressure.